Instructions

Please read the previously uploaded instructions.

Exercise 1 (4 points)

Consider the linearly independent vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$. An orthogonal set of vectors $\vec{u}_1, \dots \vec{u}_n$ can be constructed by the algorithm:

$$\begin{aligned} \vec{u}_1 &= \vec{v}_1 && (1)\\ \vec{u}_2 &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 && (2)\\ \vec{u}_3 &= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \vec{u}_2 - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 && (3)\\ \vec{u}_4 &= \vec{v}_4 - \dots && (4) \end{aligned}$$

with $\langle \cdot, \cdot \rangle$ the standard $\mathbb{R}^n$ scalar product. The general formula for $\vec{u}_k$, with $k = 1, \dots, n$, in terms of $\vec{v}_1, \dots, \vec{v}_k$ has the form: For $k = 1$

$$\vec{u}_1 = \vec{v}_1$$

For $k > 1$

$$\vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} \frac{\langle \vec{v}_k, \vec{u}_j \rangle}{\langle \vec{u}_j, \vec{u}_j \rangle} \vec{u}_j$$

(a) **(2 pts)** Write a function in Python that receives a list of 1D Numpy-arrays, where the elements of the list are $\vec{v}_1, \dots, \vec{v}_n$, and returns a list with the 1D Numpy-arrays $\vec{u}_1, \dots, \vec{u}_n$.

(b) **(2 pts)** Give the complexity of the algorithm in terms of n. Specify the constants in front of each power of n.

Exercise 2 (5 points)

Consider a matrix $A \in \mathbb{R}^{n \times n}$, a vector $\vec{b} \in \mathbb{R}^n$ and the following iterative procedure:

$$\vec{x}^{(k)} = \vec{x}^{(k-1)} + \alpha_k (\vec{b} - A\vec{x}^{(k-1)}), \alpha_k = \frac{\|\vec{b} - A\vec{x}^{(k-1)}\|^2}{(A(\vec{b} - A\vec{x}^{(k-1)}))^\intercal (\vec{b} - A\vec{x}^{(k-1)})} \quad (5)$$

1. **(0.5 pts)** Write a function that receives a matrix A and vectors $\vec{b}$ and $\vec{c}$ and returns $\vec{b} - A\vec{c}$.
2. **(0.5 pts)** Write a function that receives a vector $\vec{r}$, a matrix B and returns:

$$\frac{\|\vec{r}\|^2}{(B\vec{r})^\intercal \vec{r}}$$

3. **(2 pts)** Write a function that that receives a value $\epsilon > 0$, a vector $\vec{x}^{(0)}$ and $N \in \mathbb{N}$ and iterates according to (5) such that $\|\vec{x}^{(k)} - \vec{x}^{(k-1)}\|_2 < \epsilon$, with a maximum of N iterations, and returns $\vec{x}^{(k)}$. Use the two previously defined functions.
4. **(2 pts)** Compute the total number of operations involved in the previous question in terms of n. Assume that the maximum number of N iterations was achieved. Specify the constants in front of each power of n.